Chapter 7

Div, grad, and curl

7.1 The operator $\nabla$ and the gradient:

Recall that the gradient of a differentiable scalar field $\varphi$ on an open set $D$ in $\mathbb{R}^n$ is given by the formula:

$$\nabla \varphi = \left( \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \ldots, \frac{\partial \varphi}{\partial x_n} \right). \quad (7.1)$$

It is often convenient to define formally the differential operator in vector form as:

$$\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right). \quad (7.2)$$

Then we may view the gradient of $\varphi$, as the notation $\nabla \varphi$ suggests, as the result of multiplying the vector $\nabla$ by the scalar field $\varphi$. Note that the order of multiplication matters, i.e., $\frac{\partial \varphi}{\partial x_j}$ is not $\varphi \frac{\partial}{\partial x_j}$.

Let us now review a couple of facts about the gradient. For any $j \leq n$, $\frac{\partial \varphi}{\partial x_j}$ is identically zero on $D$ iff $\varphi(x_1, x_2, \ldots, x_n)$ is independent of $x_j$. Consequently,

$$\nabla \varphi = 0 \text{ on } D \iff \varphi = \text{constant.} \quad (7.3)$$

Moreover, for any scalar $c$, we have:

$$\nabla \varphi \text{ is normal to the level set } L_c(\varphi). \quad (7.4)$$

Thus $\nabla \varphi$ gives the direction of steepest change of $\varphi$. 
7.2 Divergence

Let \( f : \mathcal{D} \rightarrow \mathbb{R}^n, \mathcal{D} \subset \mathbb{R}^n \), be a differentiable vector field. (Note that both spaces are \( n \)-dimensional.) Let \( f_1, f_2, \ldots, f_n \) be the component (scalar) fields of \( f \). The **divergence** of \( f \) is defined to be

\[
\text{div}(f) = \nabla \cdot f = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}.
\]  
(7.5)

This can be reexpressed symbolically in terms of the dot product as

\[
\nabla \cdot f = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \cdot (f_1, \ldots, f_n).
\]  
(7.6)

Note that \( \text{div}(f) \) is a scalar field.

Given any \( n \times n \) matrix \( A = (a_{ij}) \), its **trace** is defined to be:

\[
\text{tr}(A) = \sum_{i=1}^{n} a_{ii}.
\]

Then it is easy to see that, if \( Df \) denotes the Jacobian matrix, then

\[
\nabla \cdot f = \text{tr}(Df).
\]  
(7.7)

Let \( \varphi \) be a twice differentiable scalar field. Then its **Laplacian** is defined to be

\[
\nabla^2 \varphi = \nabla \cdot (\nabla \varphi).
\]  
(7.8)

It follows from (7.1),(7.5),(7.6) that

\[
\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \cdots + \frac{\partial^2 \varphi}{\partial x_n^2}.
\]  
(7.9)

One says that \( \varphi \) is **harmonic** iff \( \nabla^2 \varphi = 0 \). Note that we can formally consider the dot product

\[
\nabla \cdot \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \cdot \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.
\]  
(7.10)

Then we have
$\nabla^2 \varphi = (\nabla \cdot \nabla) \varphi. \quad (7.11)$

Examples of harmonic functions:

(i) $D = \mathbb{R}^2$; $\varphi(x, y) = e^x \cos y$.
   Then $\frac{\partial \varphi}{\partial x} = e^x \cos y$, $\frac{\partial \varphi}{\partial y} = -e^x \sin y$,
   and $\frac{\partial^2 \varphi}{\partial x^2} = e^x \cos y$, $\frac{\partial^2 \varphi}{\partial y^2} = -e^x \cos y$. So, $\nabla^2 \varphi = 0$.

(ii) $D = \mathbb{R}^2 - \{0\}$; $\varphi(x, y) = \log(\sqrt{x^2 + y^2}) = \log(r)$.
   Then $\frac{\partial \varphi}{\partial x} = \frac{x}{x^2 + y^2}$, $\frac{\partial \varphi}{\partial y} = \frac{y}{x^2 + y^2}$,
   $\frac{\partial^2 \varphi}{\partial x^2} = \frac{(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{-2y^2}{(x^2 + y^2)^2}$, and
   $\frac{\partial^2 \varphi}{\partial y^2} = \frac{2x^2}{(x^2 + y^2)^2}$. So, $\nabla^2 \varphi = 0$.

These last two examples are special cases of the fact, which we mention
without proof, that for any function $f : D \to \mathbb{C}$ which is differentiable in the
complex sense, the real and imaginary part, $\Re(f)$ and $\Im(f)$, are harmonic
functions. Here $f$ is differentiable in the complex sense if its total derivative
$Df$ at a point $z \in D$, a priori a $\mathbb{R}$-linear map from $\mathbb{C}$ to itself, is in fact given
by multiplication with a complex number, which we then call $f'(z)$. More
concretely, this means that the matrix of $Df$ in the basis $i$ is of the form
$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for some real numbers $a, b$. We then have $f'(z) = a + bi$. There is
a large supply of such functions since any $f$ given (locally) by a convergent
power series in $z$ is complex differentiable.

In (i) we can take $f(z) = e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$ and in
(ii) we can take $f(z) = \log(z) = \log(r e^{i\theta}) = \log(r) + i\theta$ but we must be
careful about the domain. To have a well defined argument $\theta$ for all $z \in D$
we must make a "cut" in the plane and can only define $f$ on, for example,
$D = \{z = x + iy \mid y = 0 \Rightarrow x > 0\}$ or $D' = \{z = x + iy \mid y = 0 \Rightarrow x < 0\}$. But
the union of $D$ and $D'$ is $\mathbb{C} - \{0\}$ as in (ii).

(iii) $D = \mathbb{R}^n - \{0\}$; $\varphi(x_1, x_2, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\alpha/2} = r^\alpha$ for some
fixed $\alpha \in \mathbb{R}$.
   Then $\frac{\partial \varphi}{\partial x_i} = \alpha r^{\alpha-1} x_i = \alpha r^{\alpha-2} x_i$, and
   $\frac{\partial^2 \varphi}{\partial x_i^2} = \alpha (\alpha - 2) r^{\alpha-4} x_i \cdot x_i + \alpha r^{\alpha-2} \cdot 1$.
   Hence $\nabla^2 \varphi = \sum_{i=1}^n (\alpha (\alpha - 2) r^{\alpha-4} x_i^2 + \alpha r^{\alpha-2}) = \alpha (\alpha - 2 + n) r^{\alpha-2}$.
   So $\varphi$ is harmonic for $\alpha = 0$ or $\alpha = 2 - n$ ($\alpha = -1$ for $n = 3$).
7.3 Cross product in $\mathbb{R}^3$

The three-dimensional space is very special in that it admits a vector product, often called the cross product. Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the standard basis of $\mathbb{R}^3$. Then, for all pairs of vectors $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{v}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$, the cross product is defined by

$$v \times v' = \begin{vmatrix} i & j & k \\ x & y & z \\ x' & y' & z' \end{vmatrix} = (yz' - y'z)i - (xz' - x'z)j + (xy' - x'y)k. \quad (7.12)$$

**Lemma 1**

(a) $v \times v' = -v' \times v$ (anti-commutativity)  
(b) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$  
(c) $v \cdot (v \times v') = v' \cdot (v \times v') = 0$.

**Corollary:** $\mathbf{v} \times \mathbf{v} = 0$.

**Proof of Lemma**

(a) $v' \times v$ is obtained by interchanging the second and third rows of the matrix whose determinant gives $v \times v'$. Thus $v' \times v = -v \times v'$.

(b) $\mathbf{i} \times \mathbf{j} = \begin{vmatrix} 1 & j & k \\ i & y & z \\ 0 & 1 & 0 \end{vmatrix}$, which is $\mathbf{k}$ as asserted. The other two identities are similar.

(c) $v \cdot (v \times v') = x(yz' - y'z) - y(xz' - x'z) + z(xy' - x'y) = 0$. Similarly for $v' \cdot (v \times v')$.

Geometrically, $v \times v'$ can, thanks to the Lemma, be interpreted as follows. Consider the plane $P$ in $\mathbb{R}^3$ defined by $v, v'$. Then $v \times v'$ will lie along the normal line to this plane at the origin, and its orientation is given by the right hand rule: If the fingers of your right hand grab a pole and you view them from the top as a circle in the $v - v'$-plane that is oriented counterclockwise (i.e. corresponding to the ordering $(v, v')$ of the basis) then the thumb points in the direction of $v \times v'$.

Finally the length $||v \times v'||$ is equal to the area of the parallelogram spanned by $v$ and $v'$. Indeed this area is equal to the volume of the parallelepiped spanned by $v$, $v'$ and a unit vector $u = (u_x, u_y, u_z)$ orthogonal to $v$ and $v'$. We can take $u = v \times v'/||v \times v'||$ and the (signed) volume equals

$$\begin{vmatrix} u_x & u_y & u_z \\ x & y & z \\ x' & y' & z' \end{vmatrix} = u_x(yz' - y'z) - u_y(xz' - x'z) + u_z(xy' - x'y)$$

$$= ||v \times v'|| \cdot (u_x^2 + u_y^2 + u_z^2) = ||v \times v'||.$$


More generally, the same argument shows that the (signed) volume of the parallelepiped spanned by any three vectors $u,v,v'$ is $u \cdot (v \times v')$.

### 7.4 Curl of vector fields in $\mathbb{R}^3$

Let $f : \mathcal{D} \to \mathbb{R}^3$, $\mathcal{D} \subset \mathbb{R}^3$ be a differentiable vector field. Denote by $P,Q,R$ its coordinate scalar fields, so that $f = Pi + Qj + Rk$. Then the **curl of $f$** is defined to be:

$$\text{curl}(f) = \nabla \times f = \det \left( \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right). \quad (7.13)$$

Note that it makes sense to denote it $\nabla \times f$, as it is formally the cross product of $\nabla$ with $f$.

If the vector field $f$ represents the flow of a fluid, then the **curl** measures how the flow rotates the vectors, whence its name.

**Proposition 1** Let $h$ (resp. $f$) be a $C^2$ scalar (resp. vector) field. Then

(a) $\nabla \times (\nabla h) = 0$.

(b) $\nabla \cdot (\nabla \times f) = 0$.

**Proof:** (a) By definition of gradient and curl,

$$\nabla \times (\nabla h) = \det \left( \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{array} \right)$$

$$= \left( \frac{\partial^2 h}{\partial y \partial z} - \frac{\partial^2 h}{\partial z \partial y} \right) i + \left( \frac{\partial^2 h}{\partial z \partial x} - \frac{\partial^2 h}{\partial x \partial z} \right) j + \left( \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 h}{\partial y \partial x} \right) k.$$

Since $h$ is $C^2$, its second mixed partial derivatives are independent of the order in which the partial derivatives are computed. Thus, $\nabla \times (\nabla f) = 0$.

(b) By the definition of divergence and curl,

$$\nabla \cdot (\nabla \times f) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$
\[
\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial y \partial x} + \left( -\frac{\partial^2 R}{\partial y \partial z} + \frac{\partial^2 P}{\partial z \partial x}\right) + \left( \frac{\partial^2 Q}{\partial z \partial y} - \frac{\partial^2 P}{\partial x \partial z}\right).
\]

Again, since \( f \) is \( C^2 \), \( \frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 R}{\partial y \partial x} \), etc., and we get the assertion.

Done.

**Warning:** There exist twice differentiable scalar (resp. vector) fields \( h \) (resp. \( f \)), which are **not** \( C^2 \), for which (a) (resp. (b)) does **not** hold.

When the vector field \( f \) represents fluid flow, it is often called **irrotational** when its curl is 0. If this flow describes the movement of water in a stream, for example, to be irrotational means that a small boat being pulled by the flow will not rotate about its axis. We will see later in this chapter the condition \( \nabla \times f = 0 \) occurs naturally in a purely mathematical setting as well.

**Examples:** (i) Let \( D = \mathbb{R}^3 - \{0\} \) and \( f(x, y, z) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j} \). Show that \( f \) is irrotational. Indeed, by the definition of curl,

\[
\nabla \times f = \text{det} \begin{pmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{1}{x^2 + y^2} & \frac{1}{x^2 + y^2} & 0 \\
\end{pmatrix}
\]

\[
= \frac{\partial}{\partial z} \left( \frac{x}{x^2 + y^2} \right) \mathbf{i} + \frac{\partial}{\partial z} \left( \frac{y}{x^2 + y^2} \right) \mathbf{j} + \left( \frac{\partial}{\partial x} \left( \frac{-x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) \right) \mathbf{k}
\]

\[
= \left[ \frac{-(x^2 + y^2) + 2x^2}{(x^2 + y^2)^2} - \frac{-(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} \right] \mathbf{k} = 0.
\]

(ii) Let \( m \) be any integer \( \neq 3 \), \( D = \mathbb{R}^3 - \{0\} \), and \( f(x, y, z) = \frac{1}{r^m} (xi + yj + zk) \), where \( r = \sqrt{x^2 + y^2 + z^2} \). Show that \( f \) is not the curl of another vector field. Indeed, suppose \( f = \nabla \times g \). Then, since \( f \) is \( C^1 \), \( g \) will be \( C^2 \), and by the Proposition proved above, \( \nabla \cdot f = \nabla \cdot (\nabla \times g) \) would be zero. But,

\[
\nabla \cdot f = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{x}{r^m}, \frac{y}{r^m}, \frac{z}{r^m} \right)
\]

\[
= \frac{r^m - 2x^2 (m)}{r^{2m}} + \frac{r^m - 2y^2 (m)}{r^{2m}} + \frac{r^m - 2z^2 (m)}{r^{2m}}
\]

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This is non-zero as \( m \neq 3 \). So \( f \) is not a curl.

**Warning:** It may be true that the divergence of \( f \) is zero, but \( f \) is still not a curl. In fact this happens in example (ii) above if we allow \( m = 3 \). We cannot treat this case, however, without establishing Stoke’s theorem.

### 7.5 An interpretation of Green’s theorem via the curl

Recall that Green’s theorem for a plane region \( \Phi \) with boundary a piecewise \( C^1 \) Jordan curve \( C \) says that, given any \( C^1 \) vector field \( g = (P,Q) \) on an open set \( D \) containing \( \Phi \), we have:

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_C P \, dx + Q \, dy. \tag{7.14}
\]

We will now interpret the term \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \). To do that, we think of the plane as sitting in \( \mathbb{R}^3 \) as \( \{ z = 0 \} \), and define a \( C^1 \) vector field \( f \) on \( \tilde{D} := \{(x,y,z) \in \mathbb{R}^3 | (x,y) \in D \} \) by setting \( f(x,y,z) = g(x,y) = Pi + Qj \). Then

\[
\nabla \times f = \text{det} \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{pmatrix} = \left( \frac{\partial Q}{\partial z} - \frac{\partial P}{\partial x} \right) k, \text{ because } \frac{\partial P}{\partial z} = \frac{\partial Q}{\partial x} = 0.
\]

Thus we get:

\[
(\nabla \times f) \cdot k = \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}. \tag{7.15}
\]

And Green’s theorem becomes:

**Theorem 1** \( \iint_D (\nabla \times f) \cdot k \, dx \, dy = \oint_C P \, dx + Q \, dy \)

### 7.6 A criterion for being conservative via the curl

Here we just reformulate the remark after Ch. 6, Cor. 1 (which we didn’t completely prove but just made plausible) using the curl.
Proposition 1 Let \( g : \mathcal{D} \to \mathbb{R}^2, \mathcal{D} \subset \mathbb{R}^2 \) open and simply connected, \( g = (P, Q) \), be a \( C^1 \) vector field. Set \( f(x, y, z) = g(x, y) \), for all \( (x, y, z) \in \mathbb{R}^3 \) with \( (x, y) \in \mathcal{D} \). Suppose \( \nabla \times f = 0 \). Then \( g \) is conservative on \( \mathcal{D} \).

Proof: Since \( \nabla \times f = 0 \), Theorem 1 implies that \( \oint_C P \, dx + Q \, dy = 0 \) for all Jordan curves \( C \) contained in \( \mathcal{D} \). In fact, \( \nabla \times f = 0 \) also implies that \( \oint_C P \, dx + Q \, dy = 0 \) for all closed curves but we won’t prove this. Hence \( f \) is conservative. Done.

Example: \( \mathcal{D} = \mathbb{R}^2 - \{(x, 0) \in \mathbb{R}^2 \mid x \leq 0\} \), \( g(x, y) = \frac{y}{x^2+y^2} \mathbf{i} - \frac{x}{x^2+y^2} \mathbf{j} \).

Determine if \( g \) is conservative on \( \mathcal{D} \):

Again, define \( f(x, y, z) \) to be \( g(x, y) \) for all \( (x, y, z) \) in \( \mathbb{R}^3 \) such that \( (x, y) \in \mathcal{D} \). Since \( g \) is evidently \( C^1 \), \( f \) will be \( C^1 \) as well. By the Proposition above, it will suffice to check if \( f \) is irrotational, i.e., \( \nabla \times f = 0 \), on \( \mathcal{D} \times \mathbb{R} \). This was already shown in Example (i) of section 4 of this chapter. So \( g \) is conservative.